Analytic Summation.
Imaginary Sums and their Applications

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1. Introduction

The goal of this paper is to continue studying the fundamental operation of summation, the foundations for what have been laid in the previous one called **Analytic Summation.** Now, the main focus will be on extending the concept of summation on the real numbers with the emphasis on the concept of continuity.

1.1. Recap

In the previous paper, we introduced $\Delta^k, k \in \mathbb{Z}$:

Definition 1.1.1: *(Delta)*

$$\begin{aligned} \Delta a_n &\coloneqq a_{n+1} - a_n, \\ \Delta^2 a_n &\coloneqq \Delta a_{n+1} - \Delta a_n, \\ &\dots \\ \Delta^k a_n &\coloneqq \Delta^{k-1} a_{n+1} - \Delta^{k-1} a_n, \end{aligned}$$

and showed, that if a sequence $(A_n)_n \in F_\mathbb{N}$ is found such that $\Delta(A_n) = a_n$ for some sequence $(a_n)_n \in F_\mathbb{N}$, the Theorem 1.1.2 applies:

Theorem 1.1.2: *(The Fundamental Theorem Of Summation)* For all sequences $(a_n)_n, (A_n)_n \in F_\mathbb{N} : a_n = \Delta(A_n), \forall s, N \in \mathbb{N}$:

$$\sum_{n=s}^{N} a_n = A_{N+1} - A_s$$

Proof: Previous paper. ■

In this paper, we are going to extend this idea in order to learn more about the nature of the very operation of summation.

1.2. Goals

Mathematics is all about generalization. So, let us outline the goals. Namely, we want to *generalize*:

- the k-th delta of any given sequence ($k \in \mathbb{N}_0$)
- the operation of summation (extend it to (at least) the real numbers)
- the -1^{st} delta of (almost) any given sequence

And I suppose, that is already a good starting point. At least, the second point turns out to be really challenging but promising if we succeed. So, let's get started with the first one.

2. Generalizing Delta

The Definition 1.1.1 is pretty clear and obvious, but it is an inductive definition. Let us generalize it for all $k \in \mathbb{N}_0$.

2.1. One More Tool (which you can ignore for now)

Before we start, we will need one more tool. It is the well-known gamma function $\Gamma : \mathbb{C} \to \mathbb{C}$, $z \mapsto \int_0^\infty t^{z-1} e^{-t} dt$. The most important property that will be used is that it is an analytic continuation of the factorial. If you are familiar with the gamma function (which is very likely), then you can skip this part since there is nothing new to you.

Property 2.1.1: *(Gamma and Factorial)* $\forall n \in \mathbb{N} : \Gamma(n) = (n-1)!$

Property 2.1.2: *(Asymptotes of the Gamma Function)* Gamma has asymptotes at all negative integers.

2.2. A New Prospect

From now on, let F be a field and $(A_n)_n, (\alpha_n)_n \in F_{\mathbb{N}}$ sequences with $\Delta A_n = \alpha_n$

At first, we want to reformulate Definition 1.1.1 the following way:

- $k = 0$: we define $\Delta^0 \alpha_n = \alpha_n$. It makes sense considering the exponent-like properties of Δ which have been proven in the previous paper.
- $\forall k \in \mathbb{N}$: we define $\Delta^k \alpha_n = \Delta^{k-1} \alpha_{n+1} - \Delta^{k-1} \alpha_n$.

It is obvious that this it equivalent to what we had in in Definition 1.1.1. Now let us prove the following statement.

Theorem 2.2.1: $\forall k \in \mathbb{N} : \Delta^k \alpha_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^i a_{n+k-i}$.

Proof: Tedious induction, I have this written down on paper… It will later be written here, but it is a kind of Pascals Triangle. *Trust me, bro.* ■

Corollary 2.2.2: *(Generalized Form of the k-th Delta)*

$$\Delta^k \alpha_n = \sum_{i=0}^{k} \frac{\Gamma(k+1)}{\Gamma(i+1)\Gamma(k-i+1)} (-1)^i a_{n+k-i}$$

Proof: Follows directly from Theorem 2.2.1 and Property 2.1.1. ■

So, now we can just plug in whatever value of k we need and "immediately" get the formula for the k-th delta sequence. Considering Theorem 1.1.2, it is tempting to declare that we have thus found a tool for determining the exact formula of the anti-delta sequence for every given $(a_n)_n$. Unfortunately, we face a obvious problem, namely, we would then need to calculate $\sum_{k=0}^{-1} (\ldots)$ which does not make sense. Yet.

3. Summation Forms

Let us examine the basic properties of the $\sum_{n=a}^{b} (\ldots)_n$ notation. For that sake, we take an arbitrary sequence $(\alpha_n)_n \in F_{\mathbb{N}}$ and define the following map:

$$\Phi_\alpha : \mathbb{N}^2_{\leq} \to F, (a, b) \mapsto \sum_{k=a}^{b} \alpha_k,$$

where by $\mathbb{N}^2_{\leq} = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \leq b\}$. Indeed, this map is well-defined for all $a, b \in \mathbb{N} : a \leq b$. Now consider the following properties of Φ_α:

Property 3.1: Let $\alpha \in F_{\mathbb{N}}, \Phi_\alpha$ as defined above. Then we have (for all $a, b, c \in \mathbb{N} : a < b \leq c$)

(i) $\Phi_\alpha(a, a) = \alpha_a$
(ii) $\Phi_\alpha(a, b-1) + \Phi_\alpha(b, c) = \Phi_\alpha(a, c)$
(iii) $\Phi_\alpha(a, b-1) + \alpha_b = \Phi(a, b)$

Proof: Because of $a < b \leq c$, we are simply allowed to expand the sum using the definition of Φ_α. ■

Note that (i) and (ii) imply (iii), but it is an important property, so we will consider it separately.

Now, as we have outlined some of the fundamental properties of Φ_α, we can show that, in fact, these are not only *some* properties, but an equivalent definition of Φ_α. To demonstrate that, we will prove the following fact which requires the set of summation forms to be defined as follows:

Definition 3.2: We call S_α *the set of* $\mathbb{N}$*-summation forms of* α and define it as follows:

$$S_\alpha := \{\Psi_\alpha : \mathbb{N}^2_{\leq} \to F \mid \forall a, b, c \in \mathbb{N} \text{ such that } \ a < b \leq c : \\ \Psi_\alpha(a, a) = \alpha_a \text{ and } \Psi_\alpha(a, b-1) + \Psi_\alpha(b, c) = \Psi_\alpha(a, c)\}.$$

Proposition 3.3: Let $\alpha \in F_{\mathbb{N}}$, Φ_α as defined above. Then $\Psi \in S_\alpha \Rightarrow \Psi = \Phi_\alpha$. In other words, Φ_α is the only one map with the properties (i)-(iii).

Proof: Property 3.1 $\Longrightarrow \Phi_\alpha \in S_\alpha$ per definition. Now consider an arbitrary function $\Psi \in S_\alpha$ and an arbitrary fixed number $a \in \mathbb{N}$. Then we have $\Psi(a, a) = \alpha_a = \Phi_\alpha(a, a)$. Now, we will show $\forall n \in \mathbb{N}, n \geq a : \Psi(a, n) = \Phi_\alpha(a, n)$ by induction. Assume for some $n \geq a$ that $\Psi(a, n) = \Phi_\alpha(a, n)$. This implies then $\Psi(a, n+1) \overset{\text{def}}{=} \Psi(a, n) + \Psi(n+1, n+1) = \Phi_\alpha(a, n) + \alpha_{n+1} = \Phi_\alpha(a, n+1)$. Thus, we have shown that $\forall (a, b) \in \mathbb{N}^2_{\leq} : \Psi(a, b) = \Phi_\alpha(a, b) \Longleftrightarrow \Psi = \Phi_\alpha$. ■

Now, we can move on and generalize the idea of summation forms even more.

3.1. Generalized Summation Form

Now, we will temporarily go beyond the discussion of sequences and examine the properties of a general summation form defined not for a sequence, but for some function $f \in F^F$. Because of $F^{\mathbb{I}} \cong F_{\mathbb{I}}$, we can further apply what we will have learned about SF in the general case to the functions defined on a countable (or even finite) set, a.k.a. sequences.

Definition 3.1.1: *(Summation Form)* Let $f : D \to F$ be some function, whereby $D \subseteq F$ is an inductive subset of F. We call $\sigma_f : F \times F \to F$ a *summation form* if and only if, for all $x, y, z \in F$, the following conditions hold:

(i) $x \in D \Longrightarrow \sigma_f(x, x) = f(x)$
(ii) $\sigma_f(x, y-1) + \sigma_f(y, z) = \sigma_f(x, z)$.

Now leet us make the following observations:

Proposition 3.1.2: *(Reverse Sum)* For $f : D \to F$ and a summation form $\sigma := \sigma_f$

$$\forall x, y \in D : \sigma(x, y) = f(x) + f(y) - \sigma(y, x).$$

Proof: With the second property of Definition 3.1.1 we get

$$\sigma(x, y) + \sigma(y, x) = \sigma(x, y) + \sigma(y + 1, x) + \sigma(y, y) = \sigma(x, x) + \sigma(y, y) = f(x) + f(y).$$

■

Corollary 3.1.3: $\forall x \in F : \sigma(x, x - 1) = 0$

Proof: By applying Proposition 3.1.2 we get $\sigma(x, x - 1) = f(x) + f(x - 1) - \sigma(x - 1, x) = f(x) + f(x - 1) - (\sigma(x - 1, x - 1) + \sigma(x, x)) = f(x) + f(x - 1) - (f(x) + f(x - 1)) = 0$. ■

Corollary 3.1.4: For any function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$, a summation form $\sigma \coloneqq \sigma_f$ and for all $(m, n) \in \mathbb{Z}^2_{\leq}$ holds

$$\sigma(n, m) = f(m) + f(n) - \sum_{k=m}^{n} f(k),$$

which in some sense *"means"*

$$"\sum_{k=n}^{m} f(k) = "f(m) + f(n) - \sum_{k=m}^{n} f(k).$$

This *"backward sum"* with the index variable starting at a larger value than the final index will later be called an *imaginary sum*.

Now, let me explain how I perceive the concept of imaginary sums. The key to understanding the (not very obvious) validity of such an object is getting right with its name. Firstly, it is *imaginary* as it has pretty much nothing to do with the original definition of $\sum$ (which is essentially just a shorthand for writing something like $a_m + a_{m+1} + \ldots + a_n$). Nonetheless, it is a mathematically valid object, namely the value of the corresponding summation form to a sequence (which is equivalent to the only one function with a countable scope of definition). Secondly, the way it is used is very similar to that of complex, or, if you will, *imaginary* numbers. The idea of defining a "number" i such that its square is equal to -1 seems also non-related to real numbers which are, well, "real" and defined the way we would *expect* them to behave based on our observations of the universe. A lot of fairly complicated tasks stated using just the real numbers have very elegant and simple solutions involving use of complex numbers, and the general scheme looks like this:

1. We start in an already known and well-understood domain which we have a strong intuition for (e.g. the domain of real numbers, $\mathbb{R}$);
2. Then, we go beyond it to a more generalized object (e.g. $\mathbb{C}$) that in some sense *contains* our original domain (and thus the original problem). We do some magic there, involving use of some extended—*imaginary*—objects that, being (in some sense) constrained to the original scope, behave identically, but have some other nice properties that follow from their definition, allowing us to manipulate with the already existent objects on a new manner, which helps us solve the problem;
3. In the end, these *imaginary* objects yield us an answer to the original problem, "throwing us back" to the *real* domain.

And what is important, all those transitions between different domains are mathematically correct because our original domain turns out to simply be a part of that larger, generalized one (which can either "exist" or be constructed by a mathematician's imagination; the point is that it is always a kind of *extension* or *generalization*).

Corollary 3.1.5: Let $n \in \mathbb{N}$, $(\alpha_n)_n$ a sequence in $F_{\mathbb{Z}}$ (well-defined for all integers), and $\sigma := \sigma_\alpha$ its summation form. Then we have

$$"\sum_{k=0}^{-n} = "\sigma(0,-n) = -\sum_{k=1}^{n-1} \alpha_{-k}.$$

Proof: Applying Corollary 3.1.4, we get

$$\sigma(0,-n) = \alpha_0 + \alpha_{-n} - \sum_{k=-n}^{0} \alpha_k = \alpha_0 + \alpha_{-n} - \sum_{k=0}^{n} \alpha_{-k} = -\sum_{k=1}^{n-1} \alpha_{-k}.$$

■

4. Putting the Puzzle Together

Now, lets get back to the original goal: trying to generalize the process of determining a closed-form formula for calculating the sum of consecutive terms of a given sequence $(\alpha_n)_n$. Up to this point, we know three non-related things:

(i) Theorem 1.1.2 gives us a clue about how this formula should look like and behave, namely, it has to define an antiderivative sequence, or the $\Delta^{-1}\alpha_n$.

(ii) Corollary 2.2.2 tells us how to "produce" the formulas for $\Delta^k \alpha_n$ for any given $k \in \mathbb{N}_0$. Trying to extend it to negative integers fails because this would require us to somehow "calculate" $\sum_{k=0}^{-n} (...)_n$ for $n \in \mathbb{N}$, which does not make any sense.

(iii) Corollary 3.1.5 allows us to make sense of such expressions by extending the regular sum $\sum_{k=a}^{b} \alpha_k$ to the corresponding summation form $\sigma_\alpha(a,b)$, which takes the same values as the sum for all $(a,b) \in \mathbb{N}^2_{\leq}$ and is uniquely defined on all integers preserving the properties of the regular sum.

So, why not to try to combine these three facts in order to obtain a general *prescription* to constructing an anti-delta sequence?

4.1. Idea

Beforehand, we simplify the expression in Corollary 2.2.2 by defining

$$\gamma^{\alpha}_{k,i,n} := \frac{\Gamma(k+1)}{\Gamma(i+1)\Gamma(k-i+1)}(-1)^i \alpha_{n+k-i},$$

which gives us a shorter version of that formula:

$$\Delta^k \alpha_n = \sum_{i=0}^{k} \gamma^{\alpha}_{k,i,n}.$$

We extend it to all integers by replacing $\sum$ with $\sigma_\gamma := \sigma_{\left(\gamma^{\alpha}_{k,i,n}\right)_i}$:

$$\Delta^k \alpha_n = \sigma_\gamma(0,k).$$

So, we take Corollary 2.2.2, set $k = -1$, and get…

$$\Delta^{-1} \alpha_n = \sigma_\gamma(0,-1) = 0.$$

Oh, apparently, something strange is going on there. But it is actually not. It is simply how we defined it. But what if, at first, we try to obtain Δ^{-2} (with respect to n, which we denote as Δ_n) and then take the delta of that? This should be legit. Let us try to do this.

$$\Delta^{-1}_n \alpha_n = \Delta_n(\Delta^{-2}_n \alpha_n) = \Delta_n\big(\sigma_\gamma(0,-2)\big)$$

Now, we apply Corollary 3.1.5 and get

$$\Delta_n\big(-\sigma_\gamma(0,-2)\big) = \Delta_n\big(-\gamma^\alpha_{k,i,n}\big) = \Delta_n\left(-\frac{\Gamma(k+1)}{\Gamma(i+1)\Gamma(k-i+1)}(-1)^i\alpha_{n+k-i}\right) =$$
$$= \frac{\Gamma(k+1)}{\Gamma(i+1)\Gamma(k-i+1)}(-1)^{i+1}\big(\alpha_{(n+1)+k-i} - \alpha_{n+k-i}\big) \text{ for } k=-2, i=-1.$$

But here comes the problem: Γ has asymptotes in non-positive integers (see Property 2.1.2 if you are not familiar with Γ). Ideally, we should be able to set $k = -2, i = -1$ to the formula above and get

$$\Delta^{-1}\alpha_n = \frac{\Gamma(-1)}{\Gamma(0)\Gamma(0)}(\alpha_n - \alpha_{n-1}).$$

But this tiny little $\frac{\Gamma(-1)}{\Gamma(0)\Gamma(0)}$ makes it, at the first glance, impossible, because it is simply not defined. We could try to define some $g(x) := \frac{\Gamma(x-1)}{\Gamma(x)\Gamma(x)}$ and consider the limit $\lim_{x\to 0} g(x)$, but it is actually equal to zero: prove and reference this property.

What you will see from now on up until the end of this section is mathematical nonsense. But we will formalize this later. I just want to give you a clue about the motivation of doing all of this.

Let's for a moment forget about what the expression $\frac{\Gamma(-1)}{\Gamma(0)\Gamma(0)}$ really means and consider it being some *object* $\mathbb{G}$, maybe a sequence $(\mathbb{G}_n)_n$, maybe not. But suppose the formula above makes sense. What would it mean?

$$\alpha_n = \Delta_n(\Delta_n^{-1}\alpha_n) = \Delta_n(\mathbb{G}(\alpha_n - \alpha_{n-1})) = \mathbb{G}(\alpha_{n+1} - 2\alpha_n + \alpha(n-1)) \quad \text{"}\iff\text{"}$$
$$\text{"}\iff\text{"} \qquad \mathbb{G} = \frac{\alpha_n}{\alpha_{n+1} - 2\alpha_n + \alpha_{n-1}}.$$

That "means" that if $\mathbb{G}$ is actually a constant (which occurs sometimes, e. g. for $\alpha_n = b^n$ for some $b \in \mathbb{R}_{>0}$ or $\alpha_n = \sin(n)$), we can define

$$\mathbb{G}_n : F_I \to F, \alpha_n \mapsto \frac{\alpha_n}{\alpha_{n+1} - 2\alpha_n + \alpha_{n-1}} (\forall n \in I)$$

So, if for some α_n $\exists G \in F : \forall n \in I$ holds $\mathbb{G}_n(\alpha) = G$, then

$$A_n := \Delta_n^{-1}\alpha_n = G(\alpha_n - \alpha_{n-1}).$$

By applying the FTS (Theorem 1.1.2) we get for $(s, N) \in \mathbb{N}^2_\leq$:

$$\sum_{k=s}^{N} \alpha_n = A_{n+1} - A_s = G(\alpha_{n+1} - \alpha(n) - \alpha(s) + \alpha(s-1)).$$

It looks familiar, doesn't it? This is exactly the Cherry-on-Top Theorem from the previous paper. Not actually proven, but in some sense "derived"! And this is exactly how I came to this, and only then properly proved is (as you have seen in the previous paper).