Analytic Summation

Dan Demidov

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1 Introduction

1.1 Motivation

It has always been an important task to perform repetitive mathematical operations in a smart, *efficient* manner. Although it is possible to denote them using a very short and elegant notation (e.g. $\sum_{k=0}^{n} a_k$ or $\prod_{i \in I}(x-i)$), these are not very helpful at all when it comes to calculating the exact value that is represented this way because of high computational complexity which can even grow exponentially depending on the input.

One of the simplest examples of solving such a problem is the well-known formula for calculating the sum of the first n terms of the arithmetic progression:

$$\sum_{k=s}^{N} k = \frac{(N-s+1)(s+N)}{2}. \tag{1}$$

The conventional derivation process, as well as the proof that's usually conducted by induction, are incredibly easy and straightforward and will not be covered in this paper. The goal of finding such formulas is that we *significantly* reduce the amount of arithmetic operations to be performed. More formally, it can be described as that we reduce *the asymptotic time complexity* of the algorithm. Specifically, in the case of the arithmetic progression, we reduce the required time complexity from $\mathcal{O}(n)$ to $\mathcal{O}(1)$ as the required amount of arithmetical operations does not depend on the input variable n.

It is obvious that the $\sum$ notation's time complexity is always $\mathcal{O}(n)$ (linear). But what if we need to calculate something like $\sum\limits_{k=1}^{n} \frac{1}{k}$? Is there some exact formula which reduces the asymptotic time complexity, so that we do not need to sum up all of the terms "by hand"? And is it possible to generalize this for all possible sequences?

One other application of such closed formulas is the ability to extend their domain to all real (and possibly complex) numbers to better understand them.

1.2 Sequences & Notation

Before we start, I would like to clarify some notation that will be used throughout this paper, which you might already be familiar with, but which is nonetheless better to be defined to avoid ambiguity.

Notation 1.2.1. (Sequence) We will use the following notation for describing a sequence

$$(\alpha_n)_n \text{ or } (\alpha_n)_{n \in \mathbb{N}},$$

where α_n (or any other mathematical expression in those braces) is the n-th element of this sequence.

The notation $(\dots)_n$ means that the variable n is the "input index". It is important because, in more complex scenarios, it is common for sequences to be generally declared with a bunch of variables, and we use this $(\dots)_n$ notation to clarify which variable is the index-variable.

Notation 1.2.2. (The Set of Sequences)

$$F_{\mathbb{N}}$$

is the set of all sequences with the value range being some field F and the index variable from $\mathbb{N}$.

So it will be common to see something like this:

$$(\alpha_n)_n \in \mathbb{R}_{\mathbb{N}}$$

what ultimately means

$$\forall n \in \mathbb{N} : \alpha_n \in \mathbb{R}.$$

Sometimes, if it is clear from the context, we will write

$$\alpha \in F_{\mathbb{N}}, \text{ which is a shorthand for} (\alpha_n)_n \in F_{\mathbb{N}}$$

Definition 1.2.3. (Delta operator)

$$\Delta\alpha_n := \alpha_{n+1} - \alpha_n \tag{2}$$

$$\Delta^2\alpha_n := \Delta\alpha_{n+1} - \Delta\alpha_n$$

$$\dots$$

$$\Delta^{k+1}\alpha_n := \Delta^k\alpha_{n+1} - \Delta^k\alpha_n \tag{3}$$

So, Δ is an operator used to express the difference between the consecutive sequence members. Note that:

$$(\alpha_n)_n \in F_{\mathbb{N}} \implies (\Delta^k\alpha_n)_n \in F_{\mathbb{N}}, \forall k \in \mathbb{N}$$

where F is some field.

It also makes sense to define the following mapping:

Definition 1.2.4. (Delta mapping)

$$\delta : F_{\mathbb{N}} \longrightarrow F_{\mathbb{N}}, (\alpha_n)_n \mapsto (\Delta\alpha_n)_n, \tag{4}$$

where for some $k \in \mathbb{N}$

$$\delta^k : F_{\mathbb{N}} \longrightarrow F_{\mathbb{N}}, (\alpha_n)_n \mapsto (\Delta^k\alpha_n)_n. \tag{5}$$

2 Derivative of a Sequence

2.1 Motivation

All of us are (hopefully) familiar with the fundamental theorem of calculus which ultimately states that finding the area under some curve is equivalent to finding an antiderivative and calculating its values on the integration borders. More than that, integration itself is a kind of *summation*, but in the limiting case. So this begs the quation if it is possible to find a similar approach to calculating discrete partial sums? This will be discussed in more detail in the rest of the paper.

2.2 The Delta Notation

Let's discuss and prove some properties of the delta notation. For this, let $(\alpha_n)_n, (\beta_n)_n \in F_{\mathbb{N}}$.

Property 2.2.1. $\Delta^k(\Delta^l\alpha_n) = \Delta^{k+l}\alpha_n$

Proof. $\Delta^l\alpha_n = \Delta(\Delta(\dots\Delta(\alpha_n)\dots))$ where Δ is applied l times. $\Delta^k(\Delta^l\alpha_n) = \Delta(\Delta(\dots\Delta(\Delta^l\alpha_n\dots)) = \Delta(\Delta(\dots\Delta(\alpha_n)\dots))$ where Δ is applied k and then l times. □

Property 2.2.2. $\Delta(\lambda\alpha_n) = \lambda(\Delta\alpha_n)$ for $\forall\lambda \in F$

Proof. $\Delta(\lambda\alpha_n) = \lambda\alpha_{n+1} - \lambda\alpha_n = \lambda(\alpha_{n+1} - \alpha_n) = \lambda(\Delta\alpha_n)$. □ □

Property 2.2.3. $\Delta(\alpha_n + \beta_n) = \Delta\alpha_n + \Delta\beta_n$

Proof. $\Delta(\alpha_n + \beta_n) = \alpha_{n+1} + \beta_{n+1} - (\alpha_n + \beta_n) = (\alpha_{n+1} - \alpha_n) + (\beta_{n+1} - \beta_n) = \Delta\alpha_n + \Delta\beta_n$. □ □

In other words, 2.2.2 and 2.2.3 $\iff$ $\delta : F_{\mathbb{N}} \longrightarrow F_{\mathbb{N}}$, $(\alpha_n)_n \mapsto (\Delta\alpha_n)_n$ is a linear map.

Property 2.2.4. $\Delta(\frac{\alpha_n}{\beta_n}) = \frac{\Delta(\alpha_n)\beta_n - \alpha_n\Delta(\beta_n)}{\beta_n\beta_{n+1}}$

Proof. By definition $\Delta(\frac{\alpha_n}{\beta_n}) = \frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n} = \frac{\alpha_{n+1}\beta_n - \alpha_n\beta_{n+1}}{\beta_n\beta_{n+1}} = \frac{(\alpha_{n+1}-\alpha_n)\beta_n + \alpha_n\beta_n - \alpha_n\beta_{n+1}}{\beta_n\beta_{n+1}} = \frac{\Delta(\alpha_n)\beta_n - \alpha_n\Delta(\beta_n)}{\beta_n\beta_{n+1}}$. □

Property 2.2.5. $\exists C \in F : \forall n \in \mathbb{N} : \alpha_n = C \iff \Delta_n(C) = 0.$

Proof. Trivial. □

The last property motivates the following notation:

Notation 2.2.6. For some field F and inductive set I we define

$$C(F,I) := \{(\alpha_n)_n \in F_I | \Delta\alpha_n = 0 \ \forall n \in I\}$$

as the set of all constant sequences in F over I

At this moment, you might start feeling some Deja Vu. Indeed, for $der : Map(F,F) \to Map(F,F), f \mapsto f'$ it is also clear that der is a linear map, where $Map(F_1,F_2) := \{f | f : F_1 \to F_2\}$.

More than that, the delta of a fraction (2.2.4) looks very similar to the well-known formula of the derivative of a fraction. That's is another similarity that makes the connections between the two ideas even more clear and obvious.

Let's define $\delta^k((\alpha_n)_n) = (\Delta^k\alpha_n)_n$ as the **k-order derivative of a sequence** $(\alpha_n)_n$.

2.3 The Table of Basic Deltas

Lets calculate the deltas of some basic sequences to be able to use them directly if we need.

First of all, the most important (and trivial) are polynomial and exponential sequences:

$(\alpha_n)_n \in \mathbb{K}_\mathbb{N}$	$(\Delta\alpha_n)_n$	**Derivation**
$(n)_n$	1	$(n+1) - n = 1$
$(n^2)_n$	$2n+1$	$(n^2+2n+1) - n^2 = 2n+1$
...	...	...
$(n^k)_n$	$\sum_{i=0}^{k-1}\binom{k}{i}n^i$	$\sum_{i=0}^{k}\binom{k}{i}n^i - n^k = \sum_{i=0}^{k-1}\binom{k}{i}n^i$
$(2^n)_n$	2^n	$2^{n+1} - 2^n = 2^n$
$(3^n)_n$	$2 \cdot 3^n$	$3^{n+1} - 3^n = 3^n(3-1)$
...	...	...
$(b^n)_n$	$b^n(b-1)$	$b^{n+1} - b^n = b^n(b-1)$

Table 1: Deltas of Basic Sequences

One more Deja Vu is to be noted: similar to the Euler's constant $e = \lim_{n\to\inf}(1+\frac{1}{n})^n$, such that $(e^x)' = e^x$, there exists such a number in the realm of sequences (namely 2) that, raised to the n-th power, defines a sequence whose delta (a.k.a. discrete derivative) is also equal to itself. One more similarity to traditional calculus! :)

2.4 Understanding the Deja Vu

When the antiderivative of a function is defined in the context of calculus, it is common to denote it as the -1^{st} derivative, and the function itself as the 0-order derivative. Let's take a look at the current definition of delta:

$$\Delta^k\alpha_n := \Delta^{k-1}\alpha_{n+1} - \Delta^{k-1}\alpha_n, k \in \mathbb{N}.$$

It's nice to note that the expression $\Delta^0\alpha_n$ actually makes sense: Due to the similarities between the properties of exponents and the properties of Δ^k (2.2.1), the superscript k in Δ^k could be allowed to take the value 0 as well because then $\Delta^0\alpha_n = \alpha_n$ would make perfect sense with the 0-order sequence derivative being the sequence itself. That means we apply "Δ" 0 times which equates to doing nothing.

But what if we would allow... negative values for k? Let's see, what we would get if k is allowed to take value from $\mathbb{Z}$.

Suppose we have a sequence $(\alpha_n)_n \in F_{\mathbb{N}}$. Then for $k = -1$:

$$\Delta^{-1}\alpha_n = \Delta^{-2}\alpha_{n+1} - \Delta^{-2}\alpha_n.$$

Okay. But not very helpful. For $k = 0$:

$$\Delta^{0}\alpha_n = \Delta^{-1}\alpha_{n+1} - \Delta^{-1}\alpha_n.$$

And that's where it gets interesting because $\Delta^0\alpha_n = \alpha_n$. So, we have derived that, if we define $k \in \mathbb{Z}$, and set $k = 0$, it turns out that the sequence denoted as the -1-order derivative is a sequence whose derivative is the original sequence α_n. Only one problem needs to be solved. Namely, if we add some constant $C \in F$ to "the" -1-order derivative of a sequence $(\alpha_n)_n$, we get another sequence, the derivative of which is still the original sequence α_n. Indeed, suppose there is $(A_n)_n \in F_{\mathbb{N}} : \Delta A_n = \alpha_n$. Then, for all $C \in F$ 2.2.5 implies $\Delta_n(A_n + C) = \Delta_n(A_n) + \Delta_n(C) = \Delta_n(A_n) = \alpha_n$. That shows that there is no unambiguous sequence $(A_n)_n$ which can be called the -1-order derivative of $(\alpha_n)_n$. This motivates the following definition of δ^k and Δ^k for $k \in \mathbb{Z}, k < 0$:

Definition 2.4.1.

$$\delta^{-1} : F_{\mathbb{N}} \to \mathcal{P}(F_{\mathbb{N}}), (\alpha_n)_n \mapsto \{(A_n)_n \in F_{\mathbb{N}} : \Delta A_n = \alpha_n\}$$

And for $k \in \mathbb{N} \setminus \{1\}$ we define

$$\delta^{-k} : F_{\mathbb{N}} \to \mathcal{P}(F_{\mathbb{N}}), (\alpha_n)_n \mapsto \{(A_n)_n \in F_{\mathbb{N}} : \Delta A_n \in \delta^{-(k-1)}(\alpha_n)\},$$

as well as for all $k \in \mathbb{N}$:

$$\{\Delta^{-k}\alpha_n\} := \delta^{-k}(\alpha_n) \big/ C(F, I).$$

This definition assumes that $\delta^{-1}(\alpha_n) \big/ C(F, I)$ contains only one element which we call $\Delta^{-1}\alpha_n$. The following proposition shows that this is indeed true and associates a formal expression of a form $\beta_n + C(F, I) \in \delta^{-k}(\alpha_n) \big/ C(F, I)$ with a unique sequence $(\beta_n)_n$ which we define as $(\beta_n)_n =: (\Delta^{-1}\alpha_n)_n$.

Proposition 2.4.2. $|\delta^{-1}(\alpha_n) \big/ C(F, I)| = 1.$

Proof. The idea is that there is an unambiguos anti-delta sequence up to a constant, and thus, taking quotient results in a set that contains only one element (as we would simply cut out any constant term). The formal proof is also pretty easy:

Suppose that $|\delta^{-1}(\alpha_n) \big/ C(F, I)| > 1$, so $\exists A', B' \in \Delta^{-1}(\alpha_n)$ such that

$$A'_n = A_n + C(F, I) \text{ as well as } B'_n = B_n + C(F, I)$$

for some $A, B \in \delta^{-1}(\alpha_n)$. We also have for all $N \in \mathbb{N}_0$:

$$A_{N+1} - A_1 = \sum_{k=1}^{N} \alpha_k \implies A_n = \sum_{k=1}^{n-1} \alpha_k + A_1 \implies A'_n = A_n + C(F, I) = \sum_{k=1}^{n-1} \alpha_k + A_1 + C(F, I) =$$

$$= \sum_{k=1}^{n-1} \alpha_k + C(F, I) = A_n - A_1 + C(F, I) \implies A_1 = 0.$$

By analogy, we get $B_1 = 0 \implies A_1 = B_1 \implies A_n = B_n$. ☐

So now, by writing $\Delta^{-1}\alpha_n$, we will refer to this unambiguous anti-delta sequence without a constant term.

2.5 The Fundamental Theorem (of Summation)

Now, we are ready to formulate the first theorem:

Theorem 2.5.1. *$\forall (\alpha_n)_n \in F_{\mathbb{N}}$, $\forall a, b \in \mathbb{N} : b \geqslant a$*

$$\sum_{i=a}^{b} \alpha_n = \Delta^{-1}\alpha_{b+1} - \Delta^{-1}\alpha_a.$$

Proof. $a, b \in \mathbb{N}$ and $b \geqslant a \implies \exists p \in \mathbb{N}_0 : b = a + p$. Proof using induction:

$\underline{p = 0}$: $\sum\limits_{n=a}^{a+0} \alpha_n = \alpha_n = \Delta^{-1}\alpha_{a+1} - \Delta^{-1}\alpha_a \checkmark$

$\underline{\text{assume for some } p \in \mathbb{N}_0}$: $\sum\limits_{n=a}^{a+p} \alpha_n = \Delta^{-1}\alpha_{a+p+1} - \Delta^{-1}\alpha_a \iff$

$$\iff \sum_{n=a}^{a+p+1} \alpha_n = \alpha_{n+p+1} + \Delta^{-1}\alpha_{a+p+1} - \Delta^{-1}\alpha_a$$

$$\iff \sum_{n=a}^{a+p+1} \alpha_n = \Delta^{-1}\alpha_{a+p+2} - \Delta^{-1}\alpha_{a+p+1} + \Delta^{-1}\alpha_{a+p+1} - \Delta^{-1}\alpha_a$$

$$\iff \sum_{n=a}^{a+p+1} \alpha_n = \Delta^{-1}\alpha_{a+p+2} - \Delta^{-1}\alpha_a \implies \sum_{n=a}^{b} \alpha_n = \Delta^{-1}\alpha_{b+1} - \Delta^{-1}\alpha_a \text{ is correct}$$

$\forall a, b \in \mathbb{N} : b \geqslant a$.

□

What does this theorem mean? It proves and implies, that

Corollary 2.5.2. *If (for a given sequence $(\alpha_n)_n$) there exists a sequence whose delta is $(\alpha_n)_n$, it is possible to reduce the time complexity of calculating $\sum\limits_{i=a}^{b} \alpha_n$ from $\mathcal{O}(n)$ down to $\mathcal{O}(1)$.*

Proof. Theorem 1 $\implies \sum\limits_{i=a}^{b} \alpha_n = \Delta^{-1}\alpha_{b+1} - \Delta^{-1}\alpha_a$. The expression $\Delta^{-1}\alpha_{b+1} - \Delta^{-1}\alpha_a$ has a constant number of terms $\implies$ the time complexity is constant. □

2.6 Deriving First Formulas Using the Antiderivative-Sequence

Everything up to this point might seem obvious, but let's take a look at some examples that demonstrate how the theorem 2.5.1 can be applied to derive some interesting formulas (or well-known formulas but derived in a new way).

Example 2.6.1. *Consider $b \in \mathbb{R}$, $(\alpha_n)_n \in \mathbb{R}_{\mathbb{N}}$ with $\alpha_n = b^n$ $\forall n \in \mathbb{N}$.*

Let's take a look at

$$\Delta\alpha_n = b^{n+1} - b^n = b^n(b-1)$$

$$\Delta^2\alpha_n = (b-1)\Delta\alpha_n = b^n(b-1)^2$$

$$\dots$$

$$\Delta^k\alpha_n = (b-1)\Delta^{k-1}\alpha_n = b^n(b-1)^k \tag{6}$$

The latter formula (6) can easily be proven by induction for $\forall k \in \mathbb{Z}$: It is obviously true for $k = 0$ because $\Delta^0\alpha_n = \alpha_n = b^n(b-1)^0 = b^n$ ✓

Assuming that (6) is true for some $k \in \mathbb{Z}$ we easily derive both (7) and (8):

$$\Delta^{k+1}\alpha_n = \Delta^k\alpha_{n+1} - \Delta^k\alpha_n = b^{n+1}(b-1)^k - b^n(b-1)^k = b^n(b-1)^k(b-1) = b^n(b-1)^{k+1} \tag{7}$$

$$\Delta^{k-1}\alpha_n = \frac{\Delta^k\alpha_n}{b-1} = b^n(b-1)^{k-1}. \tag{8}$$

Thus, (6) $\implies \Delta^{-1}\alpha_n = b^n(b-1)^{-1} = \dfrac{b^n}{b-1}$. This and Theorem 2.5.1 imply

$$\sum_{i=0}^{n} b^i = \frac{b^{n+1}}{b-1} - \frac{b^0}{b-1} = \frac{b^{n+1}-1}{b-1}. \tag{9}$$

Which is actually the well-known formula of the sum of the first n terms of a geometric progression derived using the idea of finding a sequence whose delta is the given sequence. More than that, it also implies the formula for calculating the exact value of geometric series by taking

$$\sum_{i=0}^{\infty} b^i = \lim_{n\to\infty} \frac{b^{n+1}-1}{b-1} = \frac{1}{1-b} \text{ for } |b|< 1.$$

Example 2.6.2. *Now, let us derive the formula for calculating the sum of the first n terms of a generalized arithmetic progression.*

The first step is to consider $(\alpha_n)_n \in \mathbb{R}_{\mathbb{N}} : \alpha_n := n$. The goal now is to find a sequence $(A_n)_n \in \mathbb{R}_{\mathbb{N}}$ such that $\alpha_n = \Delta A_n$. The first step is to try to guess what this sequence might look like. The first obvious (and not very wrong) idea would be to try the sequence $A'_n = n^2$, because, as the Table 1 suggests, the degree of the polynomial is decreased by 1 when applying Δ. So, let us see, what the *discrete* derivative (a.k.a. delta) of $(A'_n)_n$ would look like:

$$\Delta A'_n = \Delta n^2 = (n+1)^2 - n^2 = 2n+1.$$

That already looks promising, doesn't it? We have just found the antiderivative-sequence of $\alpha'_n = 2n+1$. But that is not quite what we need. We need it to be just n. The next step would be to "refine" $A'_n = n^2$. Using the linearity of the delta operator, we try to (one-by-one) eliminate the "hindering" elements of the equation. First of all, lets try to get rid of that +1 term. Clearly, we need to subtract some x from n^2, such that $\Delta(n^2 - x) = 2n$. It is not hard to guess (with assistance of the Table 1), that this mysterious x should be equal to n. Indeed, using the lineairy of Δ (2.2.3), we show that

$$\Delta(n^2 - n) = \Delta(n^2) - \Delta(n) = (2n+1) - 1 = 2n.$$

The only thing left to refine is that constant factor 2. Again, due to the linearity of Δ 2.2.2:

$$\Delta(\frac{n^2-n}{2}) = \frac{1}{2}(\Delta(n^2) - \Delta(n)) = \frac{1}{2}((2n+1)-1) = n.$$

And thus, we have established the exact formula for the antiderivative-sequence $A_n = \dfrac{n^2-n}{2} = \dfrac{n(n-1)}{2}$, such that $\Delta A_n = \alpha_n$. Now, we apply the FTS 2.5.1 to calculate the sum of the first k terms of α_n:

$$\sum_{n=1}^{k} \alpha_n = A_{k+1} - A_1 = \frac{n(n+1)}{2}. \tag{10}$$

All right! That's a well known formula derived in a pretty unusual way! With a similar approach, we can derive the formula for a general arithmetic progression mentioned at the beginning (1).

Suppose we have $s \in \mathbb{N}$ (the first index to iterate from), $N \in \mathbb{N}$ (the last), and the sequence $(\alpha_n)_n \in \mathbb{R}_{\mathbb{N}} : \alpha_n = a + bn$ for some $a, b \in \mathbb{R}, n \in \mathbb{N}$. Then, to calculate $\sum\limits_{k=s}^{N} \alpha_n = \sum\limits_{k=s}^{N} a + bn$, we have to find some $(A_n)_n \in \mathbb{R}_{\mathbb{N}} : \alpha_n = \Delta(A_n)$. We could repeat ultimately the same steps we performed to derive (10), but we could do it smarter utilizing the linearity of Δ. Finding the antiderivative-sequence is the same as applying Δ^{-1} to some sequence. Since it is linear, we can simplify the current problem:

$$A_n = \Delta^{-1}(\alpha_n) = \Delta^{-1}(a+bn) = \Delta^{-1}(a) + \Delta^{-1}(bn) = a\Delta^{-1}(1) + b\Delta^{-1}(n).$$

And now, the only problem is to find the antiderivative-sequences to $(1)_n$ and $(n)_n$ which we have already done! So, we can simply substitute:

$$a\Delta^{-1}(1) + b\Delta^{-1}(n) = an + b\frac{n(n-1)}{2} = A_n$$

And thus, we have found the antiderivative-sequence for a generalized arithmetic progression. To calculate the sum, we just apply the FTS one more time:

$$\sum_{k=s}^{N} a+bn = A_{N+1}-A_s = a(N+1-s)+\frac{b}{2}(N(N+1)-s(s-1)) =$$

$$= a(N+1-s)+\frac{b}{2}(N^2+N-s^2+s+Ns-Ns) =$$

$$= a(N+1-s)+\frac{b}{2}(s(N-s+1)+N(N-s+1)) =$$

$$= a(N-s+1)+b\frac{(N-s+1)(s+N)}{2}.$$

And there we go! This is an even more generalized form of the formula mentioned at the beginning (1). There, $a=0$ and $b=1$ was assumed.

2.7 Generalized Approach?

Thus, it has become clear that the process of finding the exact formula for calculating the sum of consecutive terms of some sequence is almost the same as the process of finding the antiderivative function in order to calculate a definite integral of some function. Similarly, one has to be very patient... and, undoubtedly, very creative! Of course, some cheatsheets would be extremely useful to simplify some routine tasks, e.g. taking the anti-delta (what a name!) of some sequence defined in polynomial terms. The most obvious cheatsheet would be a some kind of an "integration table" with the antiderivatives of the trivial sequences, as well as a general formula for the anti-delta of all the sequences of the form $(n^k)_n$ for $\forall k \in \mathbb{N}$. We already have all the required tools for that, but the calculations would be pretty heavy and long. Before we start working on that formula, let us introduce some new handy methods.

3 Summation by Parts

It is often the case that a sequence (that is not obvious to find an anti-delta to) can be represented as a product of two other sequences, whose anti-deltas are already known. If this is the case when you need to integrate a function, *integration by parts* can be used to simplify the problem. But is there a similar approach if you are working with discrete sequences? And the answer is luckily YES, ABSOLUTELY.

3.1 The Approach

Theorem 3.1.1. *For all sequences* $(\alpha_n)_n, (\beta_n)_n, (A_n)_n, (B_n)_n \in \mathbb{K}_{\mathbb{N}} : \Delta(A_n) = \alpha_n$ *and* $\Delta(B_n) = \beta_n, \forall x \in \mathbb{N}:$

$$\sum_{n=1}^{x} \alpha_n\beta_n = A_{x+1}B_{x+1}+A_1B_1-A_xB_{x+1}-A_{x+1}B_x+\sum_{n=1}^{x-1}\alpha_nB_{n+1}+\sum_{n=1}^{x-1}\beta_nA_{n+1}.$$

Proof. Scary, but correct.

$$(\sum_{n=1}^{x}\alpha_n)(\sum_{n=1}^{x}\beta_n) = \sum_{n=1}^{x}\alpha_n\beta_n + \sum_{n=1}^{x}\sum_{k=n+1}^{x}(\alpha_n\beta_k+\alpha_k\beta_n) \iff$$

$$\sum_{n=1}^{x}\alpha_n\beta_n = (\sum_{n=1}^{x}\alpha_n)(\sum_{n=1}^{x}\beta_n) - \sum_{n=1}^{x}\sum_{k=n+1}^{x}(\alpha_n\beta_k+\alpha_k\beta_n) \iff$$

$$\sum_{n=1}^{x}\alpha_n\beta_n = (\sum_{n=1}^{x}\alpha_n)(\sum_{n=1}^{x}\beta_n) - \sum_{n=1}^{x}\alpha_n\sum_{k=n+1}^{x}\beta_k - \sum_{n=1}^{x}\beta_n\sum_{k=n+1}^{x}\alpha_k,$$

which, by applying the FTS 2.5.1, is equivalent to

$$\sum_{n=1}^{x}\alpha_n\beta_n = (A_{x+1}-A_1)(B_{x+1}-B_1) - \sum_{n=1}^{x}\alpha_n(B_{x+1}-B_{n+1}) - \sum_{n=1}^{x}\beta_n(A_{x+1}-A_{n+1}) \iff$$

$$\sum_{n=1}^{x} \alpha_n \beta_n = (A_{x+1} - A_1)(B_{x+1} - B_1) - \sum_{n=1}^{x-1} \alpha_n (B_{x+1} - B_{n+1}) - \sum_{n=1}^{x-1} \beta_n (A_{x+1} - A_{n+1}) \iff$$

$$\sum_{n=1}^{x} \alpha_n \beta_n = A_{x+1}B_{x+1} + A_1B_1 - A_1B_{x+1} - A_{x+1}B_1 - B_{x+1}\sum_{n=1}^{x-1} \alpha_n - A_{x+1}\sum_{n=1}^{x-1} \beta_n + \sum_{n=1}^{x-1} \alpha_n B_{n+1} + \sum_{n=1}^{x-1} \beta_n A_{n+1}.$$

By applying the FTS 2.5.1 one more time, we get

$$\sum_{n=1}^{x} \alpha_n \beta_n = A_{x+1}B_{x+1} + A_1B_1 - A_1B_{x+1} - A_{x+1}B_1 - B_{x+1}(A_x - A_1) - A_{x+1}(B_x - B_1) + \sum_{n=1}^{x-1} \alpha_n B_{n+1} + \sum_{n=1}^{x-1} \beta_n A_{n+1} \iff$$

$$\sum_{n=1}^{x} \alpha_n \beta_n = A_{x+1}B_{x+1} + A_1B_1 - A_xB_{x+1} - A_{x+1}B_x + \sum_{n=1}^{x-1} \alpha_n B_{n+1} + \sum_{n=1}^{x-1} \beta_n A_{n+1}.$$

□

At first glance, the formula may appear a bit confusing, but if α and β are *creatively* defined, some really interesting results can be obtained. Now, let us look at some of the easiest applications of the theorem 3.1.1.

Example 3.1.2. *Find the formula for calculating* $\sum_{n=0}^{N} (n \cdot 2^n)$ *for all* $N \in \mathbb{N}$ *in its closed form.*

The sequence $(n \cdot 2^n)_n$ is not in the Table 1 of Basic Sequences, and there is also no obvious formula for the anti-delta. But it can clearly be broken down into two basic sequences $\alpha_n = n$ and $\beta_n = 2^n$. We also know the anti-deltas $A_n := \Delta^{-1}(\alpha_n) = \frac{n(n-1)}{2}$ as well as $B_n := \Delta^{-1}(\beta_n) = 2^n$. We can now apply 3.1.1:

$$\sum_{n=1}^{N}(n \cdot 2^n) = \frac{N(N+1)}{2} \cdot 2^{N+1} - \frac{N(N-1)}{2} \cdot 2^{N+1} - \frac{N(N+1)}{2} \cdot 2^N + \sum_{n=1}^{N-1}(n \cdot 2^{n+1}) + \frac{1}{2}\sum_{n=1}^{N-1}(n(n+1) \cdot 2^n).$$

Oh. That last term spoils everything and makes the task even more complicated. But, as mentioned above, α and β have to be chosen CREATIVELY! Let us try to do it one more time. We notice that the last sum increases the degree of the polynomial by one. So, why not to try to choose such a sequence $(\alpha_n)_n$, so that the last sum turns out to be the one we need to express in terms of other parts of the equation, all of which will be "simpler". So we need $A_n = n - 1$. Thus, $\alpha_n = 1$, and $(\beta_n)_n$ remains unchanged.

$$\sum_{n=1}^{N}(1 \cdot 2^n) = N \cdot 2^{N+1} - N \cdot 2^{N+1} + 2^{N+1} - N \cdot 2^N + \sum_{n=1}^{N-1}(1 \cdot 2^{n+1}) + \sum_{n=1}^{N-1}(n \cdot 2^n).$$

And voila! All the terms except for $\sum_{n=1}^{N}(n \cdot 2^n)$ are known and we have closed formulas for them. Rearranging shows that

$$-(\sum_{n=1}^{N}(1 \cdot 2^{n+1}) - \sum_{n=1}^{N}(1 \cdot 2^n)) - N \cdot 2^{N+1} + N \cdot 2^{N+1} + N \cdot 2^N = \sum_{n=1}^{N-1}(n \cdot 2^n)$$

$$\iff \sum_{n=1}^{N-1}(n \cdot 2^n) = (2 - 2^{N+1}) + N \cdot 2^N \iff$$

$$\iff \sum_{n=0}^{N}(n \cdot 2^n) = 2^{N+1}(N-1) + 2.$$

Let us check the correctness of the formula by taking its delta which should be (because of 2.5.1) equal to $(n+1)2^{n+1}$:

$$\Delta(2^{n+1}(n-1)+2) = (2n \cdot 2^{n+1}+2) - (2^{n+1}(n-1)+2) = 2(n2^{n+1}) - (n2^{n+1}) + 2^{n+1} = 2^{n+1}(n+1).$$

Perfect. Let's move on.

3.2 Non-trivial applications of the Summation by Parts Rule

Now, we have the tools to define the way to derive the exact closed formula for calculating $\sigma_k(x)$ for all $k \in \mathbb{N}_0$ which we define as

Definition 3.2.1. (Polynomial Partial Sums) For $k \in \mathbb{Z}_0$ we define a map $\sigma_k : \mathbb{C} \longrightarrow \mathbb{C}$ as an extension of $\sum\limits_{i=1}^{N} n^k$ for $N \in \mathbb{N}$, such that

1. $\forall z \in \mathbb{C} : \sigma_k(z)$ is continuous,
2. $\forall N \in \mathbb{Z} \subset \mathbb{C} : \sigma_k(N) = \sum\limits_{i=1}^{N} n^k$ for $N \in \mathbb{N}$,
3. $\forall z \in \mathbb{C} : \sigma_0(z) = z$,
4. $\forall z \in \mathbb{C} : \sigma_k(z) = z^k + \sigma_k(z-1)$.
5. $\forall z \in \mathbb{C} : \sigma_k(z)$ is a polynomial.

Definition 3.2.2. For $k \in \mathbb{N}_0$ we define the sequence $(\mathbb{S}_n^k)_n \in \mathbb{C}_\mathbb{N}$:

1. $\forall k \in \mathbb{N}_0 : \mathbb{S}_1^k = 0$
2. $\forall k \in \mathbb{N}_0 : \Delta\mathbb{S}_n^k = n^k$

Proposition 3.2.3. *For* $(\mathbb{S}_n^k)_n$ *(3.2.2),* $\forall k \in \mathbb{N}_0 : \exists (b_i^k)_i \in \mathbb{Q}_\mathbb{N}$:

$$\mathbb{S}_n^k = \sum_{i=1}^{k+1} s_i^k n^i.$$

In other words, $\mathbb{S}_n^k$ *is a polynomial of degree* $k+1$.

Proof. By definition, $\Delta\mathbb{S}_n^k = n^k \implies$ repeating steps as in the Example 2.6.2 results in a closed formula for all $N \in \mathbb{N}$ for calculating $\sum_{n=1}^{N} n^k = \mathbb{S}_{N+1}^k - \mathbb{S}_1^k = \mathbb{S}_{N+1}^k$ which is a polynomial of degree $k+1$. □

Corollary 3.2.4. $\forall k \in \mathbb{N}_0 : \sigma_k(z)$ *is a polynomial of degree* $k+1$.

Proof. By definition, $\sigma_k(x)$ is a polynomial of degree $\geqslant k+1$. Definition 3.2.1 Property 4 $\implies$ $\forall x \in \mathbb{R} : f(x) - f(x-1) = k! = const$ for $f(x) := \frac{\mathrm{d}^k}{\mathrm{d}x^k}(\sigma_k(x))$. That implies $deg(f) = 1 \implies$ $deg(\sigma_k(x)) = k+1$. □

Lemma 3.2.5. $\forall k \in \mathbb{N}_0 : \sigma_k(0) = 0$.

Proof. Definition 3.2.1 Property 2 $\implies \sigma_k(1) = \sum\limits_{i=1}^{1} n^k = 1$. On the other hand, $\forall k \in \mathbb{N}_0 : \sigma_k(1) = 1^k + \sigma_k(0)$ (Definition 3.2.1 Property 4). That implies $\sigma_k(0) = 0$. □

Proposition 3.2.6. *For all* $k \in \mathbb{N}_0 : \exists!\, \sigma_k$ *as in Definition 3.2.1.*

Proof. Suppose $\exists k \in \mathbb{N} : \exists \sigma_k, \sigma_k'$ as in Definition 3.2.1, such that $\sigma_k \neq \sigma_k'$. Define $f : \mathbb{C} \longrightarrow \mathbb{C}, z \mapsto \sigma_k(z) - \sigma_k'(z)$. Property 5 implies that $f(x)$ is a polynomial as well. 3.2.4 $\implies deg(f) \leqslant k+1$. It is also true that $\forall z \in \mathbb{N} : f(z) = 0$. Hence the degree of f is finite, the infinite number of zeroes implies $f = 0$. That means that $\sigma_k = \sigma_k'$. □

Theorem 3.2.7. *(Recursive Polynomial Partial Sums) For all* $k \in \mathbb{N}_0, x \in \mathbb{C}$ *and* $(s_i^k)_i \in \mathbb{Q}_\mathbb{N}$ *(as in Proposition 3.2.3):*

$$\sigma_{k+1}(x) = \frac{1}{1 + s_{k+1}^k}(x\sigma_k(x) - \sum_{i=1}^{k} s_i^k \sigma_i(x)).$$

Proof. Let $k \in \mathbb{N}_0$. Consider sequences $\alpha_n = 1, \beta_n = n^k$ and their anti-deltas $A_n = n - 1, B_n = \mathbb{S}_n^k = \sum_{i=1}^{k+1} s_i^k n^i$. Proposition 3.2.3 implies that such s_i^k exist ($i = 1, \dots, k+1$). All the conditions for 3.1.1 are met, so we can apply the Summation by Parts rule:

$$\sum_{n=1}^{x} n^k = xB_{x+1} - (x-1)B_{x+1} - xB_x + \sum_{n=1}^{x-1} B_{x+1} + \sum_{n=1}^{x-1} n^{k+1} \iff$$

$$\iff \sigma_k(x) = x(B_{x+1} - B_x) - (x-1)B_{x+1} + \sum_{n=1}^{x} B_x + \sum_{n=1}^{x-1} n^{k+1} \iff$$

$$\iff \sigma_k(x) + (x-1)\sigma_k(x) = x^{k+1} + \sum_{n=1}^{x} B_x + \sum_{n=1}^{x-1} n^{k+1} \iff$$

$$\iff x\sigma_k(x) = \sum_{n=1}^{x}\sum_{i=1}^{k+1} s_i^k n^i + \sum_{n=1}^{x} n^{k+1} \iff x\sigma_k(x) = \sum_{i=1}^{k+1} s_i^k \sum_{n=1}^{x} n^i + \sigma_{k+1}(x) \iff$$

$$\iff \sigma_{k+1}(x) = x\sigma_k(x) - \sum_{i=1}^{k+1} s_i^k \sigma_i(x) \iff \sigma_{k+1}(x) = x\sigma_k(x) - \sum_{i=1}^{k} s_i^k \sigma_i(x) - s_{k+1}^k \sigma_{k+1}(x) \iff$$

$$\iff \sigma_{k+1}(x) = \frac{1}{1 + s_{k+1}^k}(x\sigma_k(x) - \sum_{i=1}^{k} s_i^k \sigma_i(x)).$$

□

4 The Cherry on Top

There is one more result I would really like to share. The final statement and its proof are fairly simple, but the way it has originally been derived fascinates me and leaves a lot of open questions. It will be the goal of the next paper to describe the tools, derive it as it had originally been discovered, make some additional observations, and apply the tools in a more general case. But less talking, more math:

Definition 4.0.1. We define *delta-development* of a sequence as the following family of maps $\forall n \in I$:

$$\mathbb{G}_n : F_I \to F, (a_n)_n \mapsto \frac{a_n}{a_{n+1} - 2a_n + a_{n-1}}.$$

Definition 4.0.2. For an inductive subset I we define $I^- := \{n \in I \mid n - 1 \in I\}$

Theorem 4.0.3. (The Cherry-on-Top Theorem) *Let $(a_n)_n \in F_{I^-}$. If there exists a number $G \in F$, such that, for all $n \in I^- : \mathbb{G}_n = G$, then an anti-delta sequence $(A_n)_n \in F_{I^-}$ of $(a_n)_n$ is explicitly given as the following expression in closed form:*

$$A_n = G(a_n - a_{n-1}),$$

and the sum of consecutive terms of a_n starting at some index $s \in I^-$ up to $N \in I^-(N \geqslant s)$ is also explicitly given in closed form:

$$\sum_{k=s}^{N} a_n = A_{N+1} - A_s = G(a_{N+1} - a_N - a_s + a_{s-1}).$$

Proof. Suppose $\exists G \in F$, such that $\forall n \in I^- : \mathbb{G}_n = G$. Then define a sequence $A_n := G(a_n - a_{n-1})$. We show that $\Delta A_n = a_n$, which would prove the theorem (by FTS 2.5.1).

$$\forall n \in I^- : \Delta A_n = G\Delta(a_n - a_{n-1}) = G(a_{n+1} - 2a_n + a_{n-1}) =$$

$$= \mathbb{G}_n(a_{n+1} - 2a_n + a_{n-1}) = \frac{a_n}{a_{n+1} - 2a_n + a_{n-1}}(a_{n+1} - 2a_n + a_{n-1}) = a_n.$$

□

And this turns out to be useful and powerful. Let's look at some examples.

Example 4.0.4. *Geometric progression. Once again.*

Set $a_n = b^n$ for some $b \in \mathbb{R}, n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$:

$$\mathbb{G}_n((a_n)_n) = \frac{b^n}{b^{n+1} - 2b^n + b^{n-1}} = \frac{1}{b - 2 + b^{-1}} = const,$$

which implies that the cherry-on-top theorem (COTT) applies, and for $N \in \mathbb{N}$ we get

$$\sum_{n=0}^{N} a_n = \frac{1}{b-2+b^{-1}}(b^{N+1} - b^N - b^0 + b^{-1}) = \frac{b^N(b-1) - 1 + b^{-1}}{b - 1 - (1 - b^{-1})} = \frac{(b-1)(b^N - b^{-1})}{(b-1)(1-b^{-1})}$$

$$= \frac{b^{N+1} - 1}{b - 1}.$$

Alright, this works. But now, I want to show you another example, which is not so trivial.

Example 4.0.5. $a_n := \sin(n), n \in \mathbb{N}$. *Then for all* $n \in \mathbb{N}$:

$$\mathbb{G}_n((a_n)_n) = \frac{\sin(n)}{\sin(n+1) - 2\sin(n) + \sin(n-1)} =$$

$$= \frac{\sin(n)}{\sin(n)\cos(1) + \sin(1)\cos(n) - 2\sin(n) + \sin(n)\cos(1) - \sin(1)\cos(n)} =$$

$$= \frac{1}{2(\cos(1) - 1)} = const.$$

$$\implies \sum_{n=0}^{N} \sin(n) = \frac{\sin(N+1) - \sin(N) + \sin(-1)}{2(\cos(1) - 1)}.$$

4.1 Questions

The importance of $\mathbb{G}_n$ is already made clear. But, up to this point, only the case $\mathbb{G}_n(a) = const$ has been made useful. The question is, whether it is possible to "extract" more information from the delta-development of a sequence and use it to find closed-form solutions to $\Delta^{-1}a_n$. The motivation for delta-development, its original "derivation", as well as the questions about other cases, are going to be the main focus of the next paper on this topic. If you have any ideas and/or suggestions, I would really like to cooperate to uncover the mysteries of sequences and summation.